Chapter 16 – Complex Numbers

16.1 Basic Relationships

The Cartesian form of a complex number is $a + jb$, where $j = \sqrt{-1}$. The complex number $a + jb$ is represented on the complex plane by the vector OP, for P with coordinates (a, b).

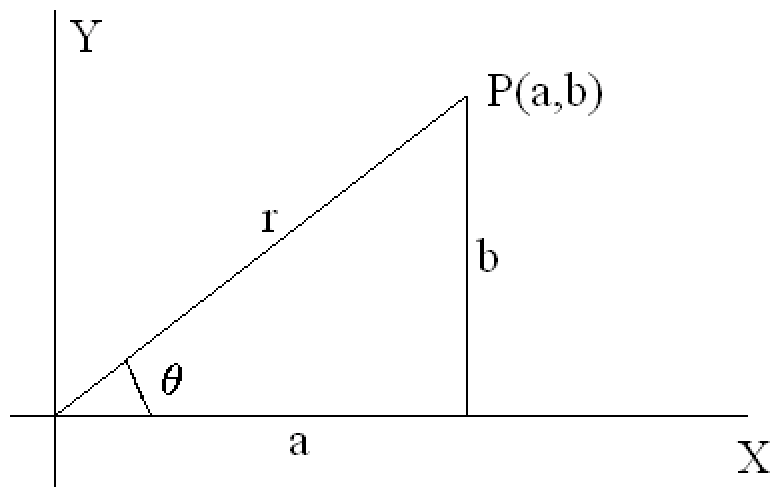

Fig 16.1

On occasions it is necessary to convert to Cartesian form to the polar form (i.e. $r(\cos\theta + j\sin\theta)$ and vise versa. The polar form can also be expressed as $re^{j\theta}$ by Euler's Equation. Often we express $r(\cos\theta + j\sin\theta)$ as $r\angle\theta$ for brevity.
From Fig. 16.1 it is clear that the following relationships connect the polar and Cartesian forms –

$$a = r\cos\theta$$
$$b = r\sin\theta$$
$$r = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}\frac{b}{a}$$

(the last coming from $\tan\theta = \dfrac{b}{a}$).

16.2 Converting Cartesian Form to Polar

To convert $a + jb$ to $r\angle\theta$ we use $\tan\theta = \dfrac{b}{a}$ (i.e. $\theta = \tan^{-1}\dfrac{b}{a}$) to find θ, and $r = \dfrac{b}{\sin\theta}$ (from $\sin\theta = \dfrac{b}{r}$) to find r.

Note: when 'a' and/or 'b' are negative, this means the complex number lies in the 2nd, 3rd, or 4th quadrant.
The angle θ is thus affected, but not the amplitude, r. Hence, for placing a and b on the Slide Rule, we take their absolute values (i.e. $|a|$ and $|b|$).
If φ is the angle obtained in any of the methods above (using the absolute values of a and b) then for the various quadrants θ is obtained by –

a) Second Quadrant (a<0, b>0)
   θ = 180° - φ
b) Third Quadrant (a<0, b<0)
   θ = 180° + φ

c) Forth Quadrant (a>0, b<0)

$\theta = 360° - \varphi$

A. For S, T_1 and T_2 scales on the body of the Slide Rule.

Example 1: Convert 4 + 3j to polar form:

1. Set the hair line over 3 on the D scale.
2. Place the left index of the CI scale under the hair line. (in some cases the right index)
3. Reset the hair line over 4 on the CI scale.
4. Under the hair line read off 36.85° on the T_1 scale as the value for θ. (Use T_1 scale if $\frac{b}{a} < 1$ and T_2 scale if $\frac{b}{a} > 1$.)
5. Reset the hair line over 36.85° on the S scale.
6. Under the hair line read off 5 on the CI scale as the value for r.

$$\therefore 4 + 3j = 5\angle 36.85°$$

B. For S and T scales on the slide and a DI scale there are two cases.

For θ < 45° (i.e. $\frac{b}{a} < 1$)

Example 2: Convert 4 + 3j to polar form.

1. Set the hair line over the left index of the DI scale.
2. Place the 3 of the C scale under the hair line.
3. Reset the hair line over the 4 on the DI scale.
4. Under the hair line read off 36.85° on the T scale as the value for θ.
5. Reset the hair line over 36.85° on the S scale as the value for θ.
6. Under the hair line read off 5 on the DI scale as the value for r.

$$\therefore 4 + 3j = 5\angle 36.85°$$

For θ > 45° (i.e. $\frac{b}{a} > 1$)

Example 3: Convert 3+4j to polar form.

1. Set the hair line over the left index of the DI scale.
2. Place the 3 of the C scale under the hair line. (Note, we have ‘3’ here as the value of ‘a’, in contrast to the ‘3’ in step 2 of Example 2 which was then the value for ‘b’).
3. Reset the hair line over the 4 on the DI scale.
4. Under the hair line read off 36.85° on the T scale, so that θ = 90° - 36.85° = 53.15°.
5. Reset the hair line over 36.85° on the S scale as the value for θ.
6. Under the hair line read off 5 on the DI scale as the value for r.

$$\therefore 3 + 4j = 5\angle 53.15°$$

Note: These two cases can be brought into one general method by using first the C scale, whichever of a and b is the smaller. Then if $\frac{b}{a} < 1$, the angle is taken as read off the T scale, otherwise for $\frac{b}{a} > 1$, we take the complement of the angle found on the T scale.

C. For S and T scales on the slide and no DI scale, there are two cases:

For θ < 45° (i.e. $\frac{b}{a} < 1$)

Example 4: Convert 4 + 3j to polar form.

1. Set the hair line over 4 on the D scale.
2. Place the right index of the C scale under the hair line.
3. Reset the hair line over 3 on the D scale.
4. Under the hair line read off 36.85° on the T scale as the value for θ.
5. Place the 36.85° on the S scale under the hair line.
6. Below the right index of the C scale read off 5 on the D scale as the value for r.

$$\therefore 4+3j=5\angle 36.85°$$

For θ > 45° (i.e. $\frac{b}{a}>1$)

Example 5: 3 + 4j to polar form.

1. Set the hair line over 4 on the D scale.
2. Place the right index of the C scale under the hair line.
3. Reset the hair line over 3 on the D scale.
4. Under the hair line read off 36.85° on the T scale so that θ = 90° - 36.85° = 53.15°
5. Reset the hair line over 36.85° on the S.
6. Under the hair line read off 5 on the D scale as the value for r.

$$\therefore 3+4j=5\angle 53.15°$$

Exercise 16(a)

Convert the following to polar form:

(i) 8 + 6j =
(ii) 5 + 12j =
(iii) 12 + 5j =
(iv) -3 + 4j =
(v) -41 – 11j =
(vi) 34 – 7.2j =
(vii) -7 + 25j =
(viii) 2 – 3j

16.3 Converting Polar Form to Cartesian

To convert $r\angle\theta$ to a + jb we use b = r sin θ to find b and a = $\frac{b}{\tan\theta}$ (from $\tan\theta=\frac{b}{a}$) to find a.

Note: for angles, θ, greater than 90° (that is complex numbers in the 2nd, 3rd, or 4th quadrant) we express the angle as φ (for φ<90°) by –

d) Second Quadrant (90°a<θ<180°)
   φ = 180° - θ
e) Third Quadrant (180°a<θ<270°)
   φ = θ – 180°
f) Forth Quadrant (270°a<θ<360°)
   φ = 360° - θ

A. For S, T_1 and T_2 scales on the body of the Slide Rule.

Example 1: Convert $13\angle 59°$ to Cartesian form:

1. Set the hair line over 59° on the S scale.
2. Place the 14 of the CI scale under the hair line.
3. Reset the hair line over the index of the CI scale.
4. Under the hair line read off 11.15 on the D scale as the value for b.
5. Reset the hair line over 59° on the T_2 scale. (Use the T_1 scale if θ < 45°)
6. Under the hair line read off 6.7 on the CI scale as the value for a.

$$\therefore 13\angle 59° = 6.7+11.15j$$

B. For S and T scales on the slide and a DI scale there are two cases.

For θ < 45°

Example 2: Convert $13\angle 31°$ to Cartesian form.

1. Set the hair line over 13 on the DI scale.
2. Place the 31° of the S scale under the hair line.
3. Reset the hair line over the right index of the DI scale.
4. Under the hair line read off 11.15 on the C scale as the value for a.
5. Reset the hair line over 31° on the T scale.
6. Under the hair line read off 6.7 on the DI scale as the value for b.

   $\therefore 13\angle 31^\circ = 11.15 + 6.7j$

Example 1: Convert $13\angle 59^\circ$ to Cartesian form:
For the complement of 59° = 90° - 59° = 31°

1. Set the hair line over 13° on the DI scale.
2. Place the 31° of the S scale under the hair line.
3. Reset the hair line over the index of the DI scale.
4. Under the hair line read off 11.15 on the C scale as the value for b.
5. Reset the hair line over 31° on the T scale.
6. Under the hair line read off 6.7 on the DI scale as the value for a.

   $\therefore 13\angle 59^\circ = 6.7 + 11.15j$

Note: These two cases can be brought into one general method by using the angle as given, if it is less than 45°, otherwise we use its complement. If the angle is less than 45°, we read ‘a’ off the C scale and ‘b’ off the DI scale. If the angle given is greater than 45°, we read ‘b’ off the C scale and ‘a’ off the DI scale.

C. For S and T scales on the slide and no DI scale.

To convert $13\angle 31^\circ$ to Cartesian form, we evaluate –

a = 13 cos 31° = 11.5
and b = 13 sin 31° = 6.7
to obtain 11.15 + 6.7j

Exercise 16(b)

Convert the following to polar form:

(i) $4\angle 60^\circ$
(ii) $4\angle 30^\circ$
(iii) $6.5\angle 42^\circ$
(iv) $6.5\angle 132^\circ$
(v) $10.6\angle 216^\circ$
(vi) $34\angle 304^\circ$
(vii) $105\angle 110^\circ$
(viii) $15\angle 143.15^\circ$

16.4 Miscellaneous Problems

Recall:
$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r\ \times r_2\angle\theta_1 + \theta_2$$
$$r_1\angle\theta_1 \div r_2\angle\theta_2 = r\ \div r_2\angle\theta_1 - \theta_2$$

Exercise 16(c)

Express the answer to the following in polar form:

(i) $35\angle 21^\circ \times 19\angle 53^\circ$
(ii) $4.7\angle 34^\circ \div 8.6\angle 41^\circ$
(iii) $(4+3j)\times(2+3j)$
(iv) $(6+9j)\div(5+3j)$

Express the answer to the following in Cartesian form:

(v) $2.5\angle 133^\circ \times 6.88\angle 68^\circ$
(vi) $42\angle 110^\circ \div 72\angle 140^\circ$
(vii) $(3+37)\times(-2+5j)$
(viii) $(-2-2j)\div(3-4j)$